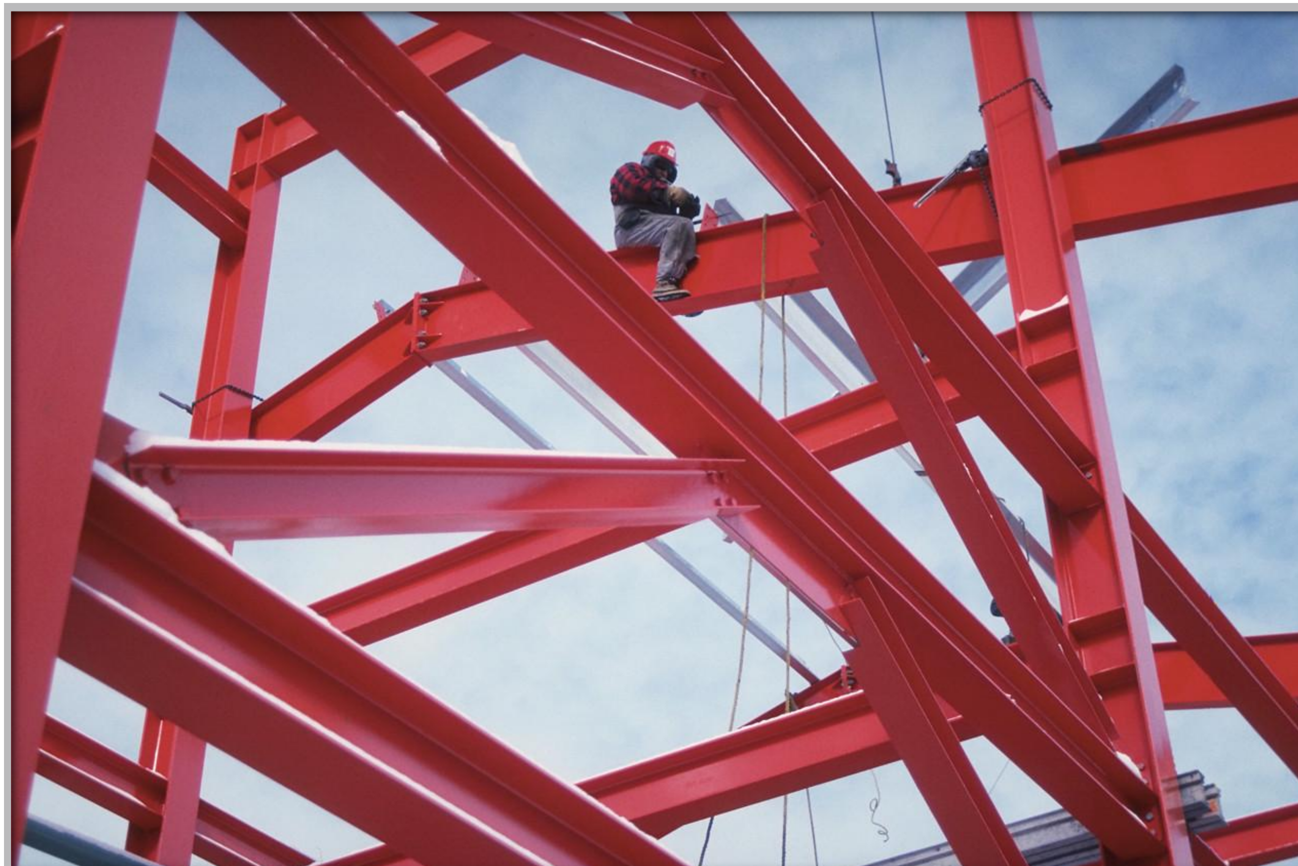
Mechanical Vibrations

VIBRATIONS

Single DoF free vibration system



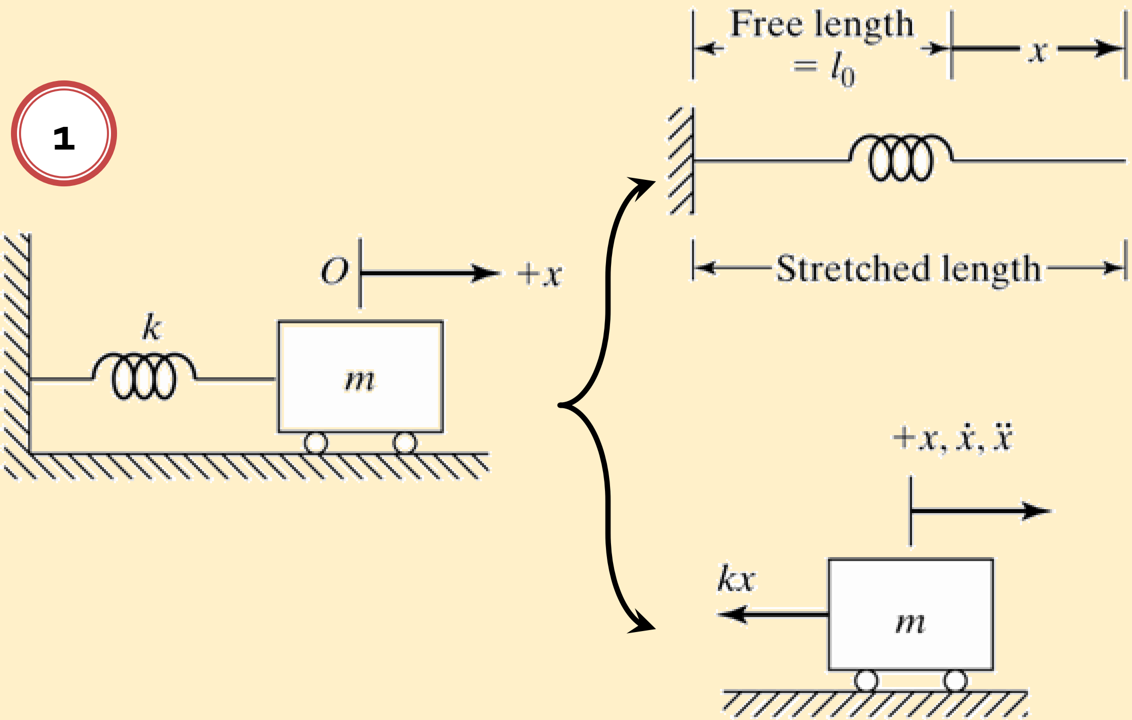
Eng. Laith Batarseh

Single DoF free vibration system

VIBRATIONS

Introduction

1



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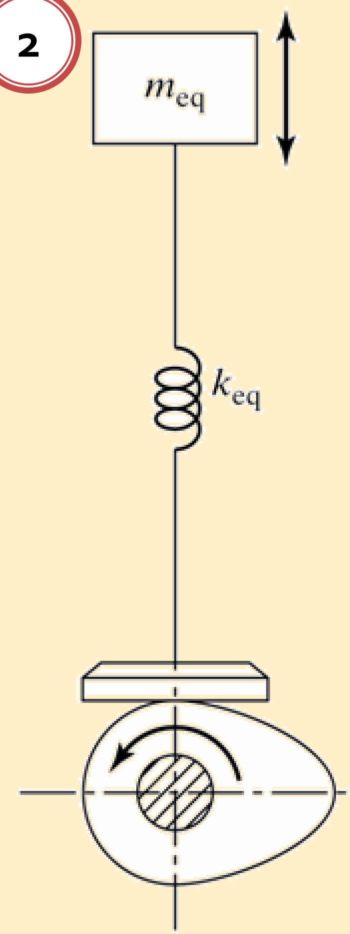
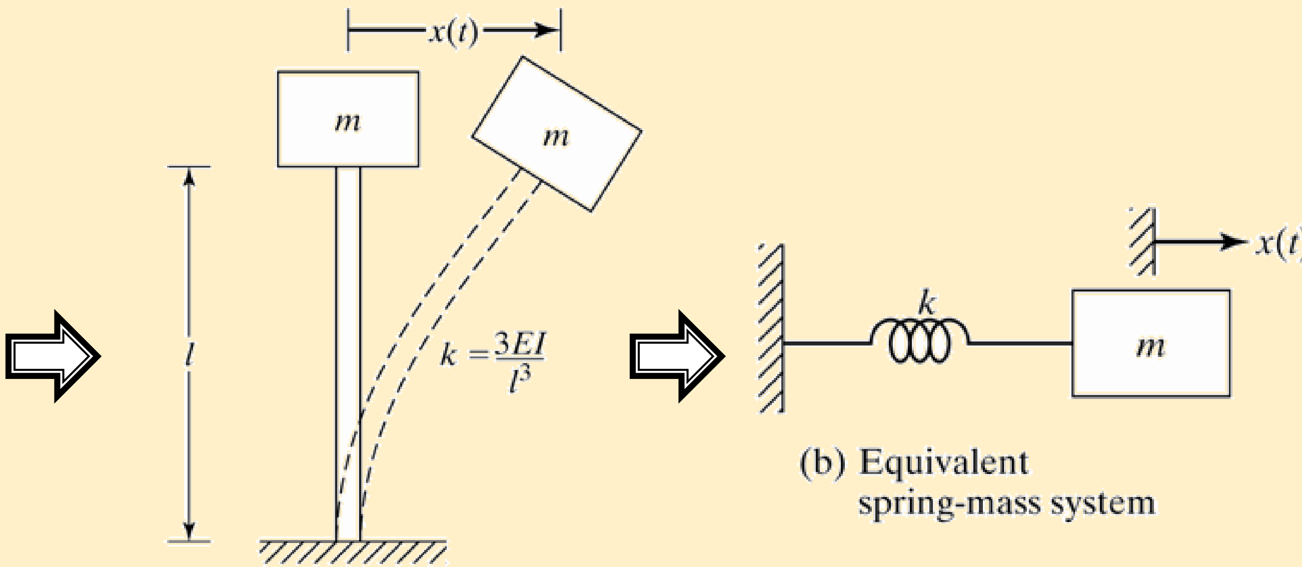


Single DoF free vibration system

Examples

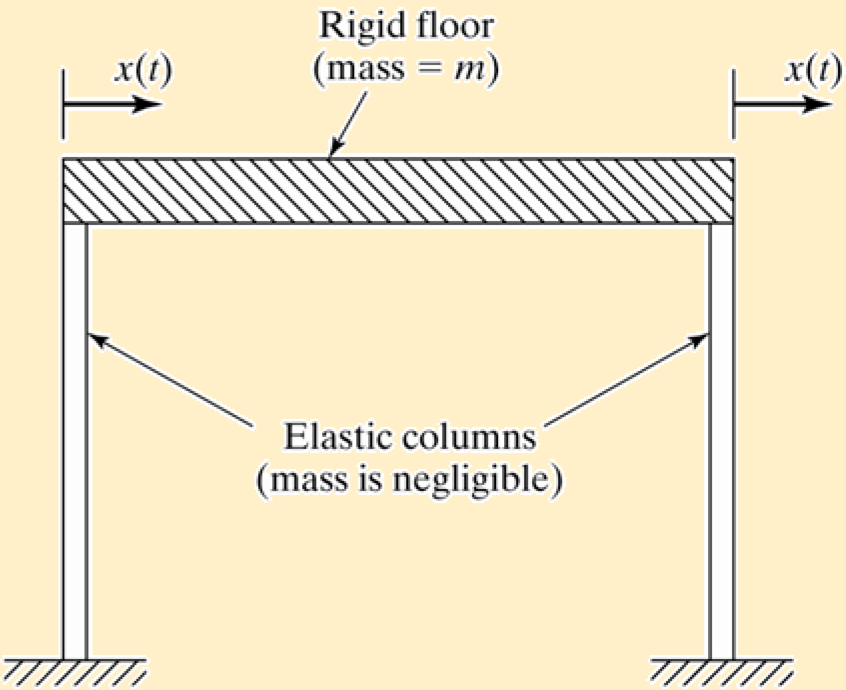


(a) Idealization of the tall structure

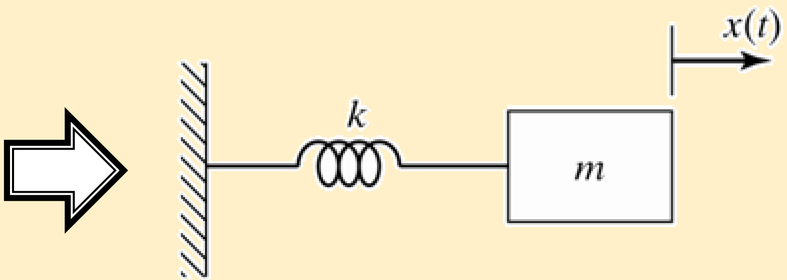
(b) Equivalent spring-mass system

Single DoF free vibration system

❑ Examples

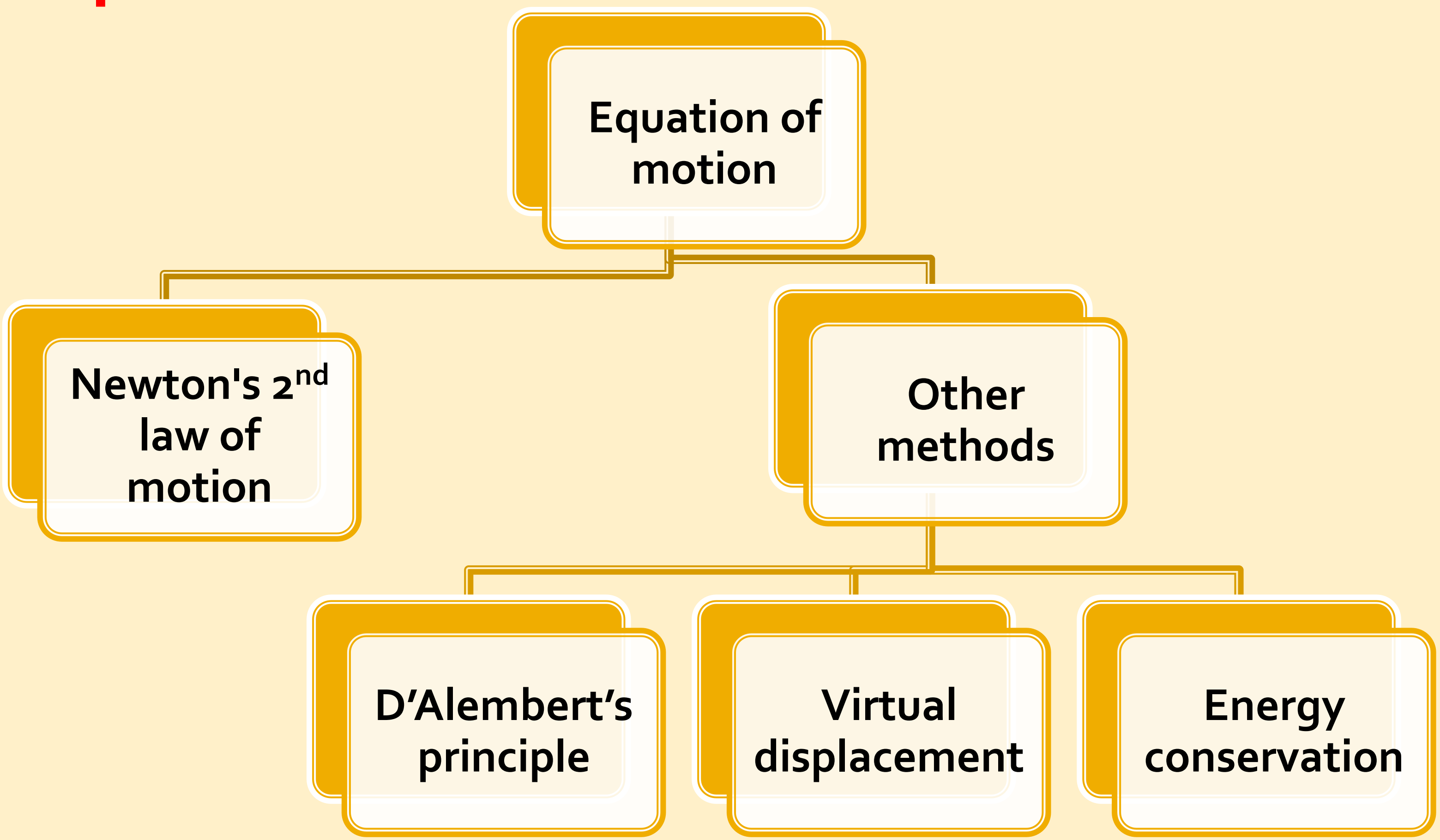


(a) Building frame



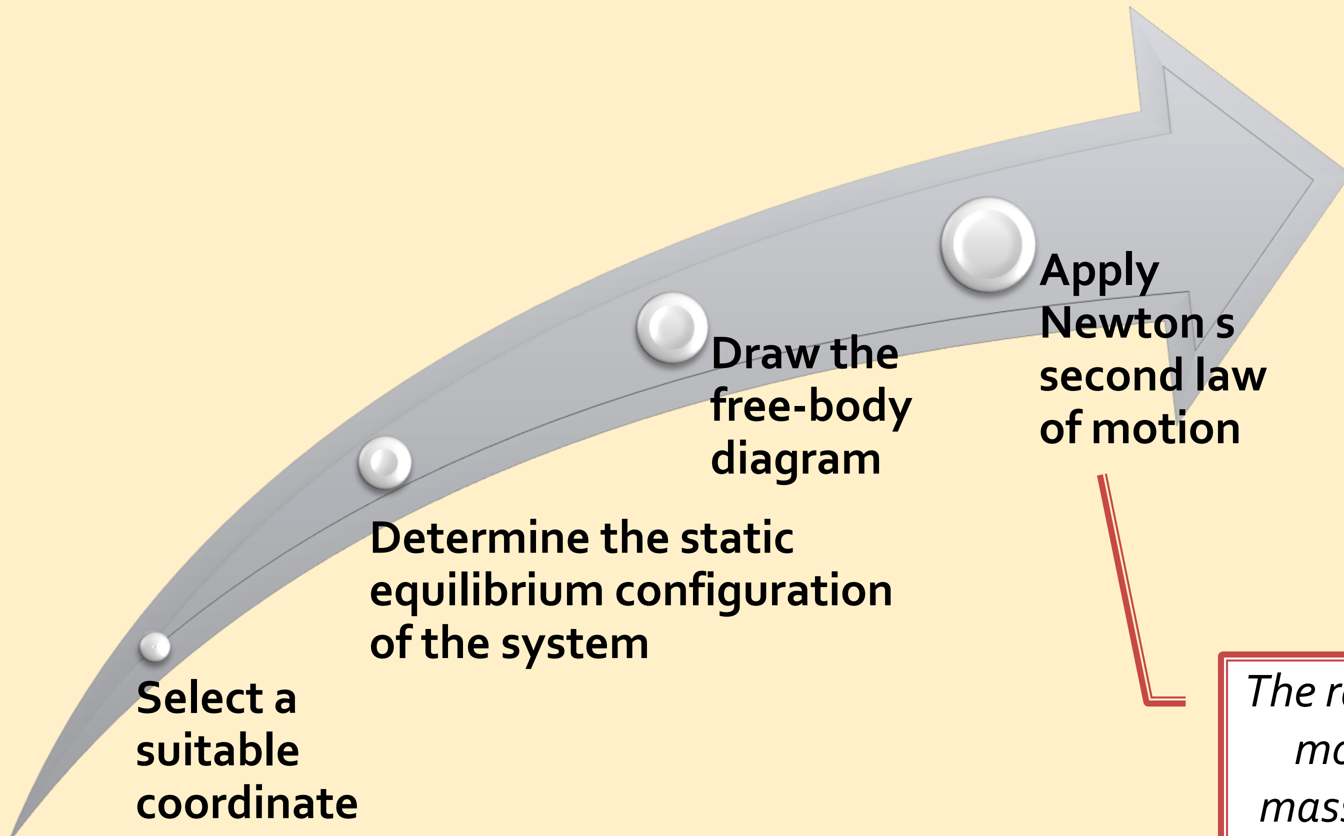
Single DoF free vibration system

Equation of motion



Single DoF free vibration system

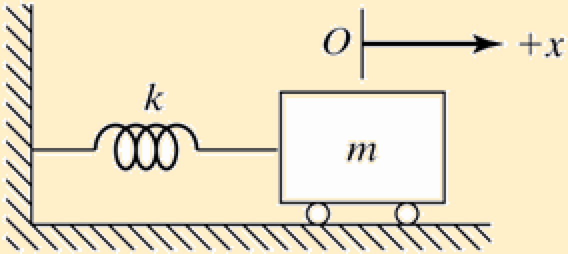
❑ Newton's 2nd law of motion



The rate of change of momentum of a mass is equal to the force acting on it.

Single DoF free vibration system

Newton's 2nd law of motion



$$\vec{F}(t)=\frac{d}{dt}\left(m\frac{d\vec{x}(t)}{dt}\right)=m\frac{d^{2}\vec{x}(t)}{dt^{2}}=m\ddot{x}$$

$$\vec{F}(t)=-kx=m\ddot{x}\Rightarrow \boxed{m\ddot{x}+kx=0}$$

Energy conservation

Kinetic energy

$$T=\frac{1}{2}m\dot{x}^{2}$$

Potential energy

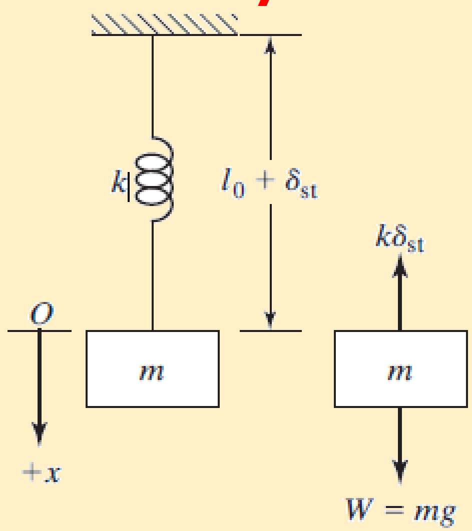
$$U=\frac{1}{2}kx^{2}$$

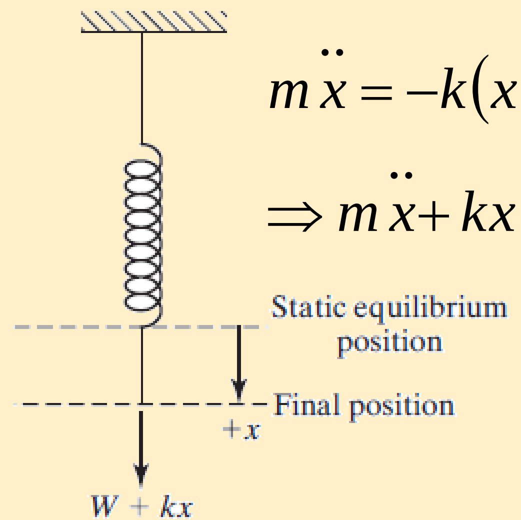
$$T+U=constant$$

$$\frac{d}{dt}\left(T+U\right)=0 \quad\Longrightarrow\quad m\ddot{x}+kx=0$$

Single DoF free vibration system

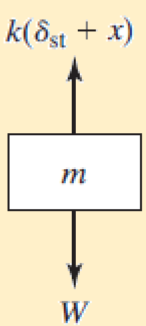
Vertical system

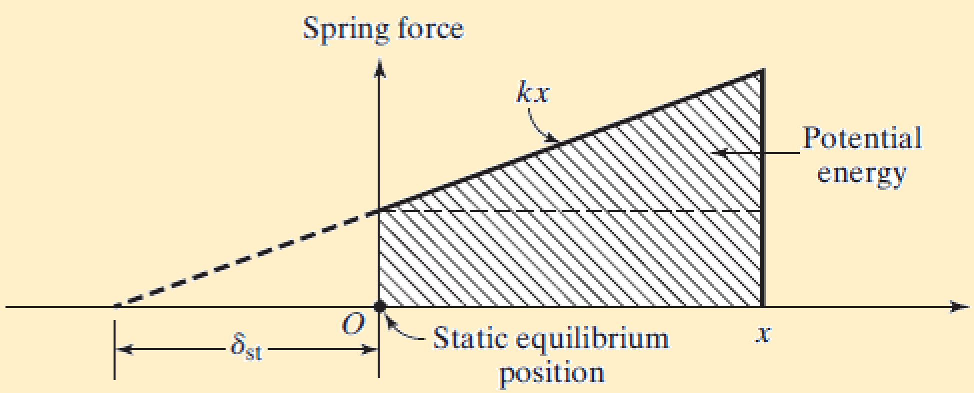




$$m\ddot{x} = -k(x+\delta_{st}) + mg \;\; ; mg = k\delta_{st}$$

$$\Rightarrow m\ddot{x} + kx = 0$$





Single DoF free vibration system

Mathematical review

$$A_1(t)\ddot{x}(t)+A_2(t)\dot{x}(t)+A_3(t)x(t)=F(t)$$

$F(t)=0 \rightarrow$ Homogenous

$x(t)=C_1x_1+C_2x_2$

If A_1, A_2 and A_3 are constants:
Characteristic equation:
$A_1s^2+A_2s+A_3=0$

$$s=\frac{-A_2\pm\sqrt{A_2^2-4A_1A_3}}{2A_1}$$

$F(t)\neq 0 \rightarrow$ none homogenous

$x(t)=C_1x_1+C_2x_2+x_p$

x_p form is the same type as $F(t)$

Single DoF free vibration system

$$s = \frac{-A_2 \pm \sqrt{A_2^2 - 4A_1A_3}}{2A_1}$$

Only one real root:

$s_{1,2}$

$x(t) = C_1e^{st} + C_2\,t\,e^{st}$

Two unequal real root: $s_{1,2}$

$x(t) = C_1e^{s_1t} + C_2\,e^{s_2t}$

Complex root:

$s_{1,2} = \alpha \pm \beta i$

$x(t) = e^{\alpha t}\,\{C_1\cos(\beta t) + C_1\sin(\beta t)\}$

Single DoF free vibration system

VIBRATIONS

❑ Solution

- $m\ddot{x}+kx=0$ **is 2nd order homogenous differential equation**

where: A_1 = m, A_2=0 and A_3=k

Natural frequency

$$s=\frac{\pm\sqrt{-4mk}}{2m}=\pm i\sqrt{\frac{k}{m}}=\pm i\omega_n$$

- **the solution** : $x(t) = e^{\alpha t}\{C_1\cos(\beta t) + C_1\sin(\beta t)\}$

  where $\alpha = 0$ and $\beta = \omega_n$

- $x(t) = C_1\cos(\omega_n t) + C_2\sin(\omega_n t)$

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- **Solution**
- **C_1 and C_2 can be determined from initial conditions (I.Cs). For this case we need two I.Cs.**

$$x(t=0)=C_1=x_o$$

$$\dot{x}(t=0)=C_2\omega_n=\dot{x}_o$$

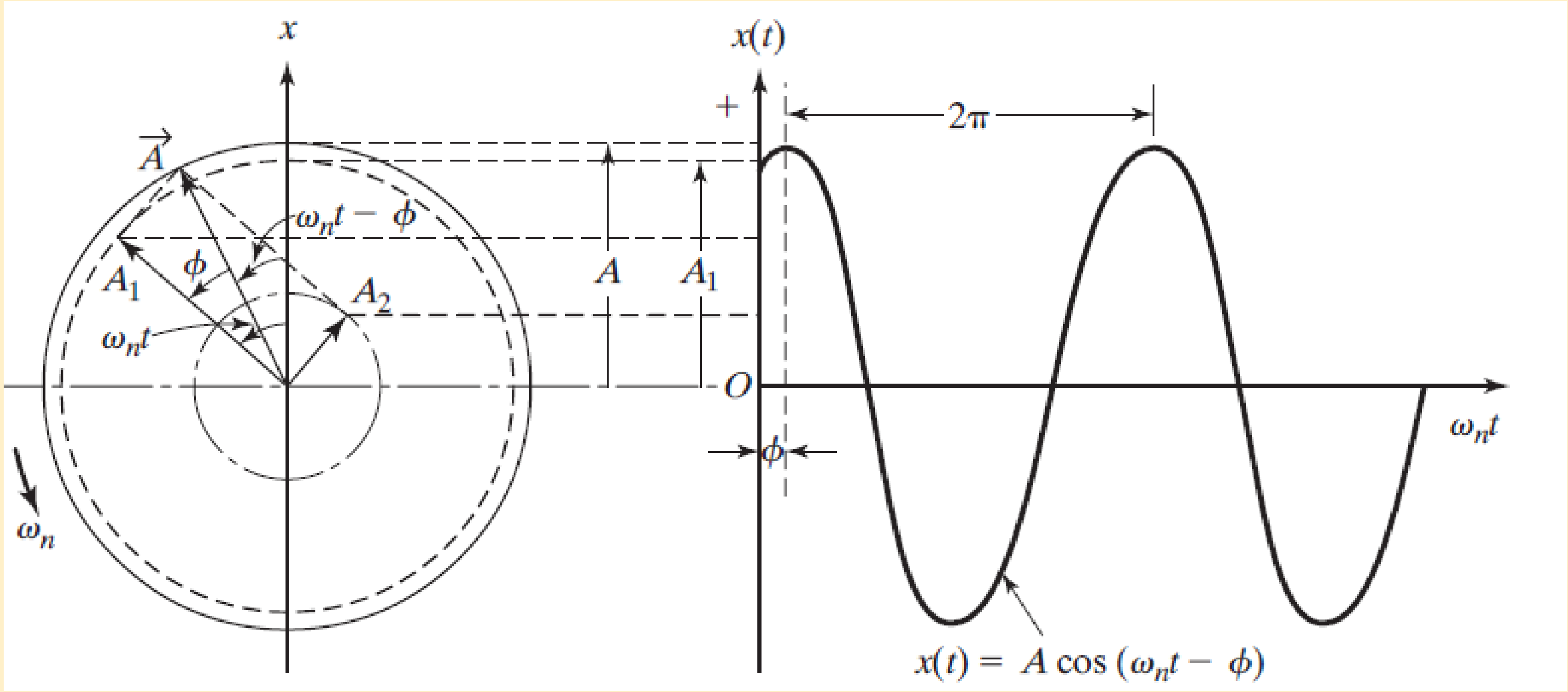
- **The I.Cs for this case would be:**
- **So:** $x(t)=x_o\cos(\omega_n t)+\dfrac{\dot{x}_o}{\omega_n}\sin(\omega_n t)---Eq.1$

Single DoF free vibration system

VIBRATIONS

- **Harmonic motion**

Introduce Eq.2 into Eq.1: $x(t) = A\cos(\omega_n t - \phi) = A_o \sin(\omega_n t + \phi_o)$



Single DoF free vibration system

Harmonic motion

$$x(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$$

$$x(t) = x_o \cos(\omega_n t) + \frac{\dot{x}_o}{\omega_n} \sin(\omega_n t)$$

Harmonic functions in time.

Mass spring system is called harmonic oscillator

Assume:

C_1 = A cos(ϕ) --- Eq.2(a)

C_2 = A sin(ϕ) --- Eq.2(b)

$$A = \sqrt{C_1^2 + C_2^2} = \sqrt{x_o^2 + \left(\frac{\dot{x}_o}{\omega_n}\right)^2} = Amplitude$$

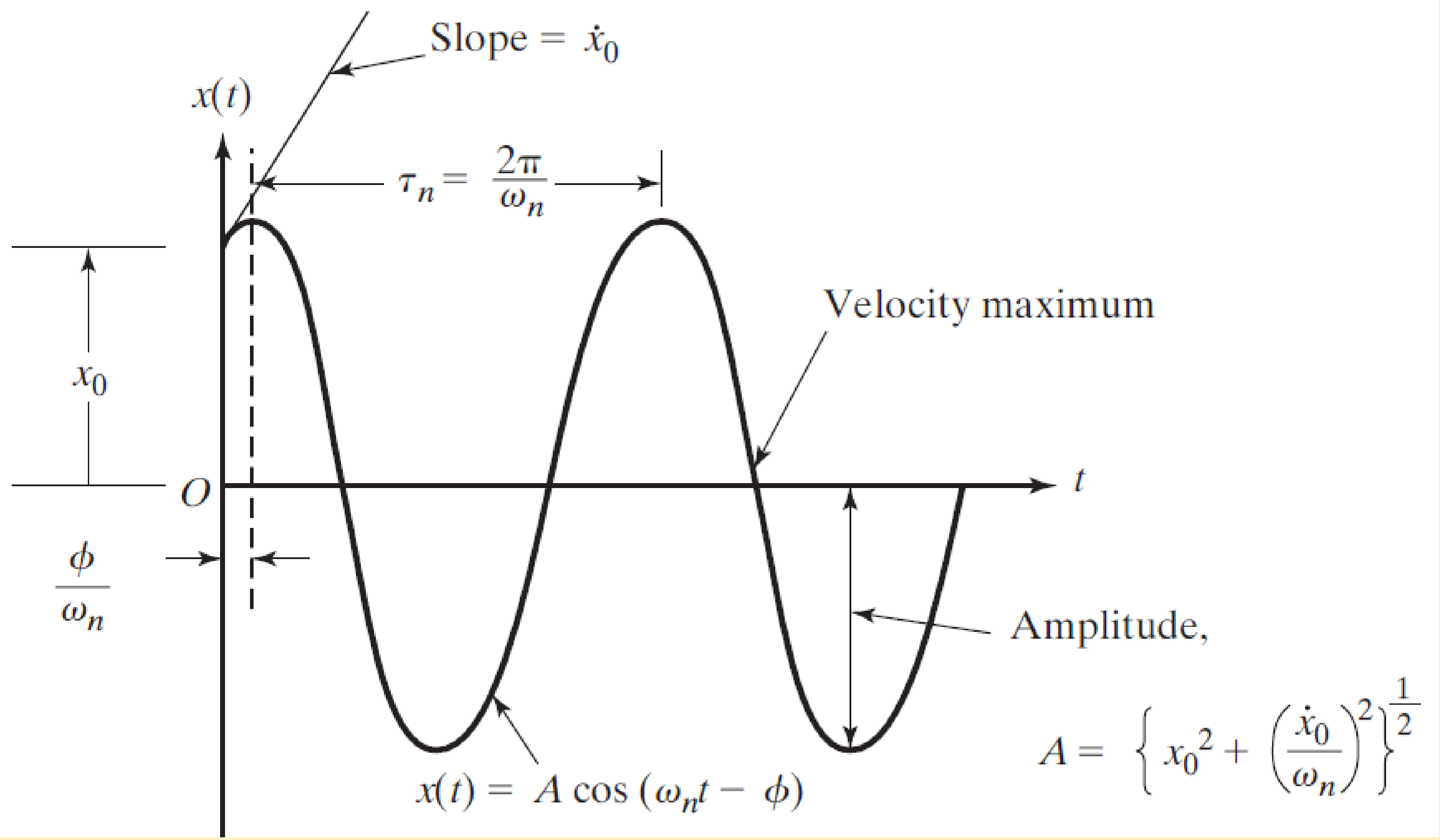
$$\phi = \tan^{-1}\left(\frac{C_2}{C_1}\right) = \tan^{-1}\left(\frac{\dot{x}_o}{x_o \omega_n}\right) = \text{Phase angle}$$

$$A_o = A$$

$$\phi_o = \tan^{-1}\left(\frac{x_o \omega_n}{\dot{x}_o}\right)$$

Single DoF free vibration system

- **Harmonic motion**



Single DoF free vibration system

- **Natural Frequency (N.F)**

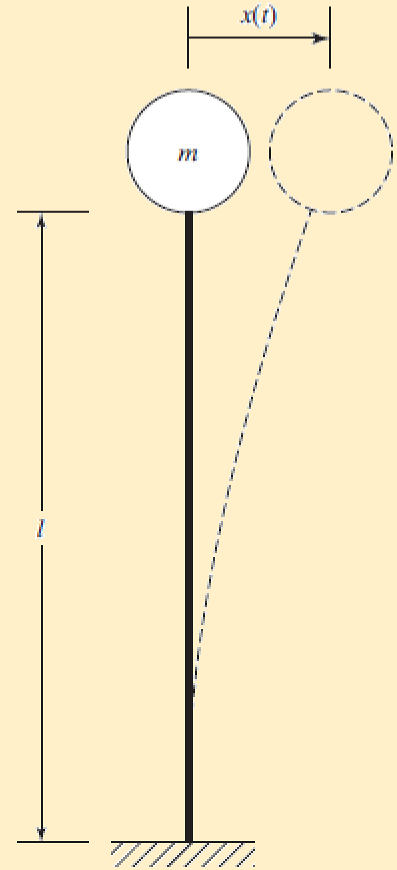
- **A system property (i.e. depends of system parameters *m* and *k*)**
- **Unit: rad/sec**
- **It is related to the periodic time (τ) : $\tau = 2\pi/\omega_n$**
- **Periodic time is the time taken to complete one cycle (i.e. 4 strokes)**
- **The relation between the ω_n and τ is inverse relation**

Single DoF free vibration system

Example 2.1

The column of the water tank shown in Fig is 90m high and is made of reinforced concrete with a tubular cross section of inner diameter 2.4m and outer diameter 3m. The tank mass equal 3 x 10^5 kg when filled with water. By neglecting the mass of the column and assuming the Young's modulus of reinforced concrete as 30 Gpa. determine the following:

- **the natural frequency and the natural time period of transverse vibration of the water tank**
- **the vibration response of the water tank due to an initial transverse displacement of 0.3m.**
- **the maximum values of the velocity and acceleration experienced by the tank.**



Single DoF free vibration system

VIBRATIONS

Example 2.1 solution:

Initial assumptions:

1. **the water tank is a point mass**
2. **the column has a uniform cross section**
3. **the mass of the column is negligible**
4. **the initial velocity of the water tank equal zero**

Single DoF free vibration system

Example 2.1 solution:

a. Calculation of natural frequency:

1. Stiffness: $k = \frac{3EI}{l^3}$ **But:** $I = \frac{\pi}{64}\left(d_o^4 - d_i^4\right) = \frac{\pi}{64}\left(3^4 - 2.4^4\right) = 2.3475\, m^4$

So: $k = \frac{3x30x10^9\, x2.3475}{90^3} = 289{,}812\; N/m$

2. Natural frequency : $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{289{,}812}{3x10^5}} = 0.9829\, \text{rad}/s$

Single DoF free vibration system

Example 2.1 solution:

b. Finding the response:

1. $x(t) = A\sin(\omega_n t + \phi)$

$$A = \sqrt{x_o^2 + \left(\frac{\dot{x}_o}{\omega_n}\right)^2} = x_o = 0.3m \qquad \phi = \tan^{-1}\left(\frac{x_o\omega_n}{\dot{x}_o}\right) = \tan^{-1}\left(\frac{x_o\omega_n}{0}\right) = \frac{\pi}{2}$$

So, $x(t) = 0.3\sin(0.9829t + 0.5\pi)$

Single DoF free vibration system

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Example 2.1 solution:

c. Finding the max velocity:

$$\dot{x}(t) = 0.3(0.9829)\cos\left(0.9829t + \frac{\pi}{2}\right) \Rightarrow \dot{x}_{\max} = 0.3(0.9829) = 0.2949 m/s$$

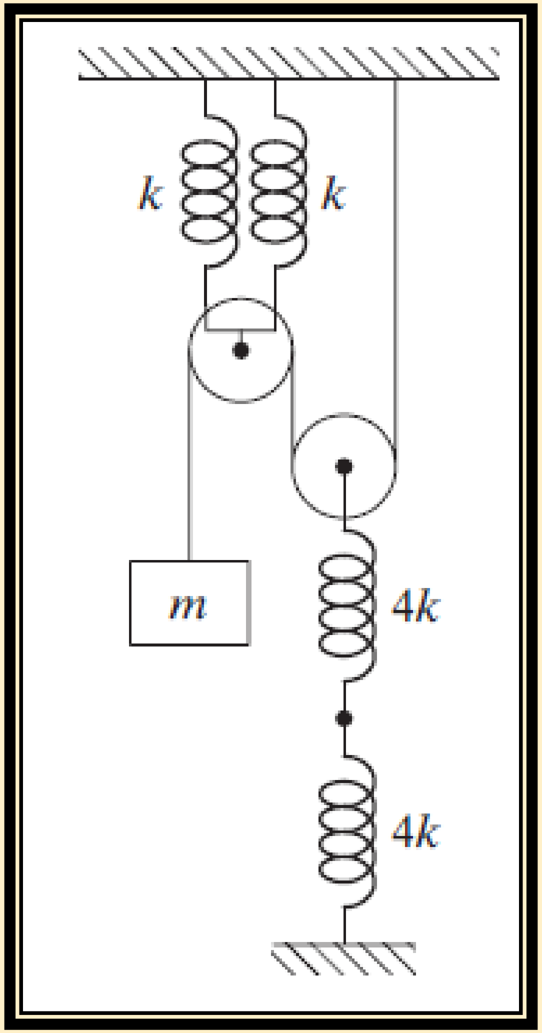
Finding the max acceleration :

$$\ddot{x}(t) = -0.3(0.9829)^2 \sin\left(0.9829t + \frac{\pi}{2}\right) \Rightarrow \ddot{x}_{\max} = 0.3(0.9829)^2 = 0.2898 m/s^2$$

Single DoF free vibration system

❑**Example2 : Q2.13**

Find the natural frequency of the pulley system shown in Fig. by neglecting the friction and the masses of the pulleys.



Single DoF free vibration system

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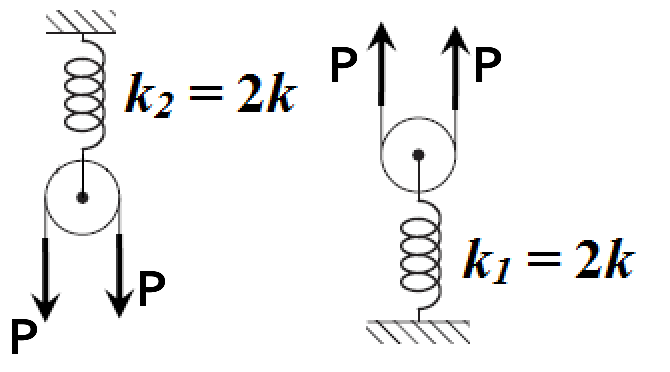
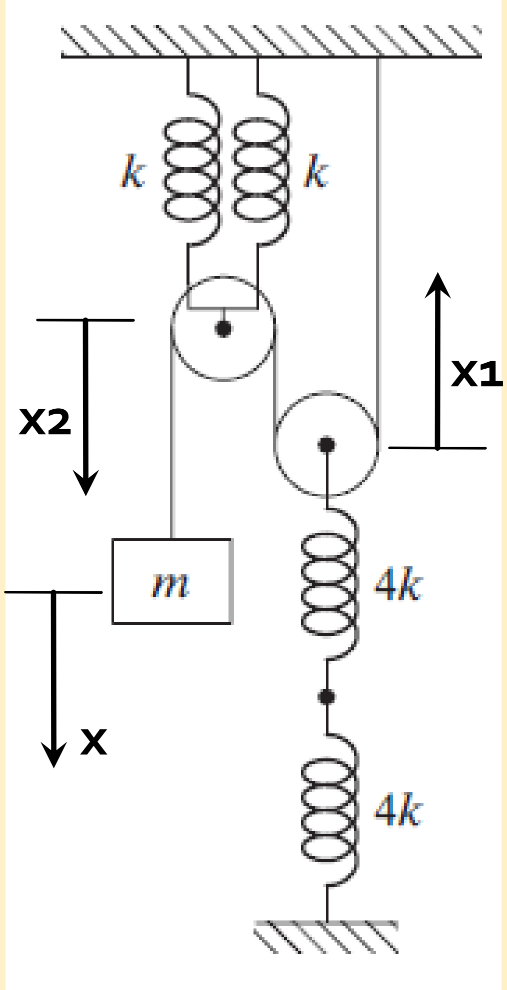
- Example2 : Q2.13

Solution:

1. Free body diagram





2. $x = 2x_1 + 2x_2$ ---- Eq.1

Single DoF free vibration system

Example2 : Q2.13

Solution:

3. Equilibrium for pulley_1 : $2P = k_1 x_1 = 2k\, x_1$ ---- Eq.2

4. Equilibrium for pulley_2 : $2P = k_2 x_2 = 2k\, x_2$ ---- Eq.3

5. Substitute Eqs 2 and 3 in Eq.1: $x = 2\left(\frac{2P}{k_1}\right) + 2\left(\frac{2P}{k_2}\right) = 4P\left(\frac{1}{2k} + \frac{1}{2k}\right) = \frac{4P}{k}$

6.Let k_{eq} is the equivalent spring constant for the system: $k_{eq} = \frac{P}{x} = \frac{k}{4}$

7. Mathematical model: $m\ddot{x} + kx = 0$

8. Natural frequency: $\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{4m}}$

Single DoF free vibration system

VIBRATIONS

❑ **Rotational system**

Governing equation:

$$\sum M_O = J_O \alpha = J_O \ddot{\theta}$$

$$J_O \ddot{\theta} + mgl \sin(\theta) = 0$$

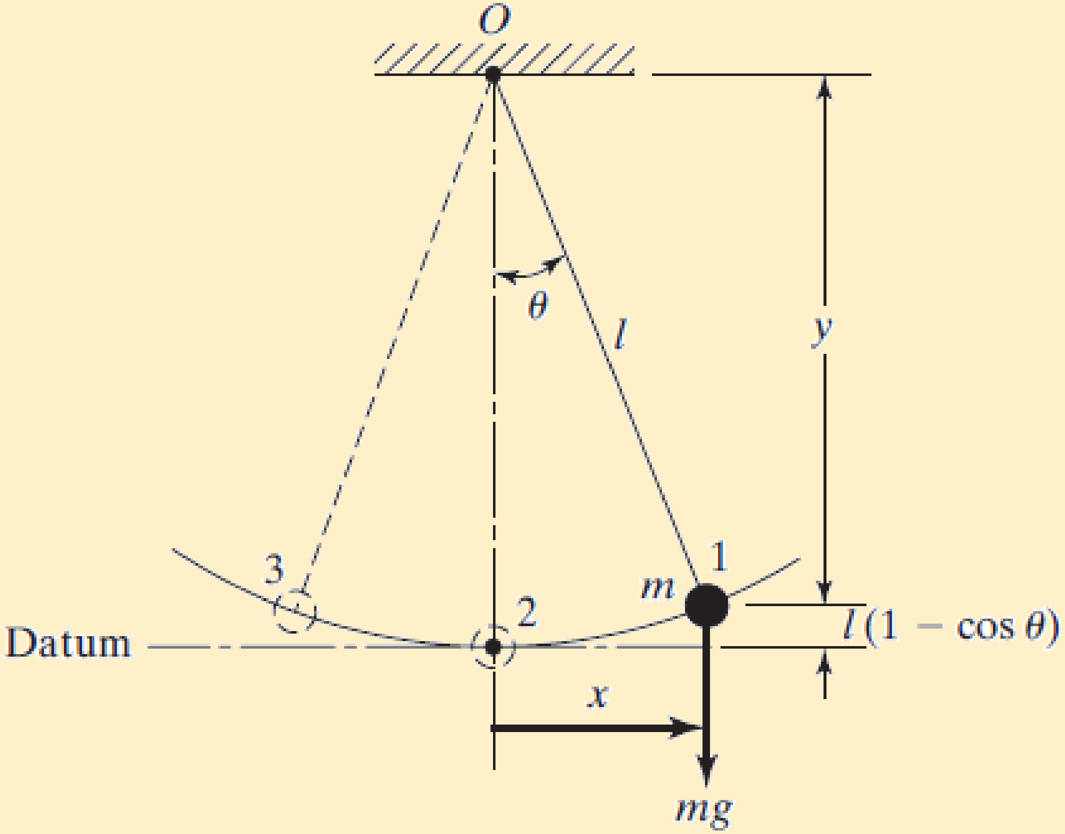
Assume θ is very small

$$\sin(\theta) \cong \theta$$

$$J_O \ddot{\theta} + (mgl)\theta = 0$$

Natural frequency (ω_n)

$$\omega_n = \sqrt{\frac{mgl}{J_O}} \Rightarrow \tau = 2\pi \sqrt{\frac{J_O}{mgl}}$$



Single DoF free vibration system

Torsional system

Governing equation:

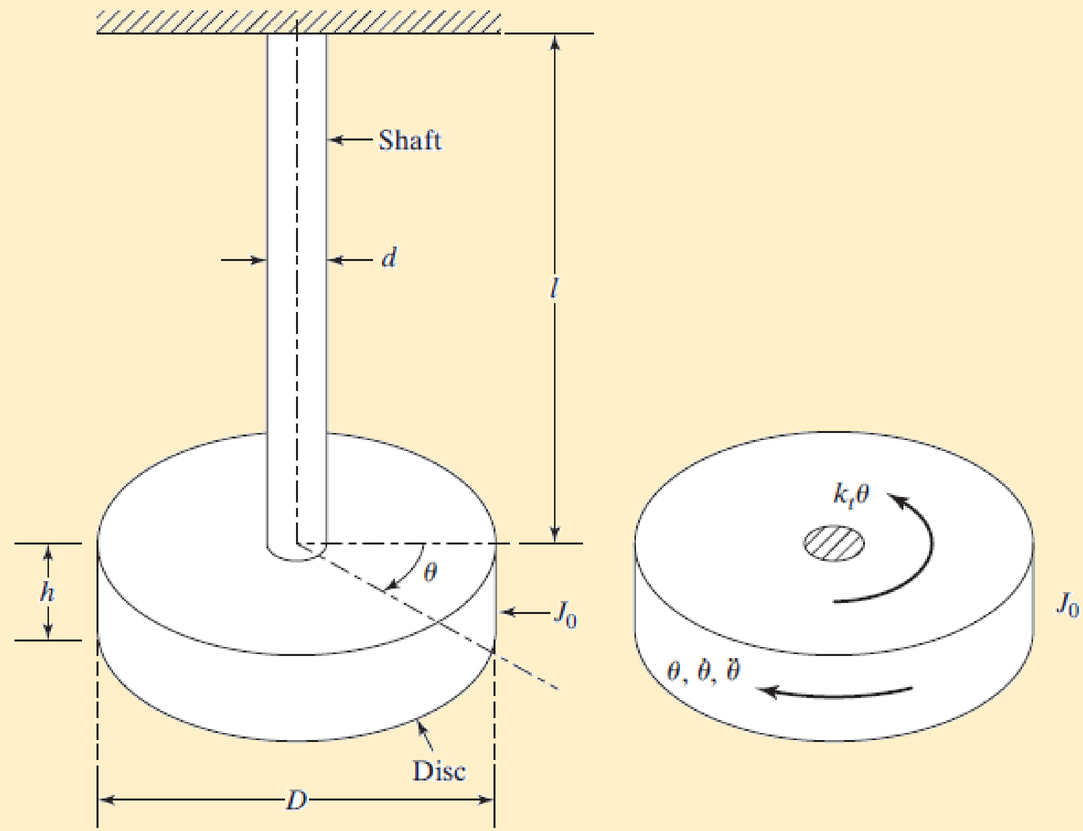
$$\sum M_O = J_O \alpha = J_O \ddot{\theta}$$

$$J_O \ddot{\theta} + k_T \theta = 0$$

Natural frequency (ω_n)

$$\omega_n = \sqrt{\frac{k_T}{J_O}} \Rightarrow \tau = 2\pi \sqrt{\frac{J_O}{k_T}}$$

$$J_O = \frac{\rho h \pi D^4}{32} = \frac{W D^2}{8g}$$



Single DoF free vibration system

VIBRATIONS

- **Solution**

$$\theta(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t)$$

$$\theta(t=0) = A_1 = \theta_o$$

$$\dot{\theta}(t=0) = A_2 \omega_n = \dot{\theta}_o$$

$$\theta(t) = \theta_o \cos(\omega_n t) + \frac{\dot{\theta}_o}{\omega_n} \sin(\omega_n t)$$

Single DoF free vibration system

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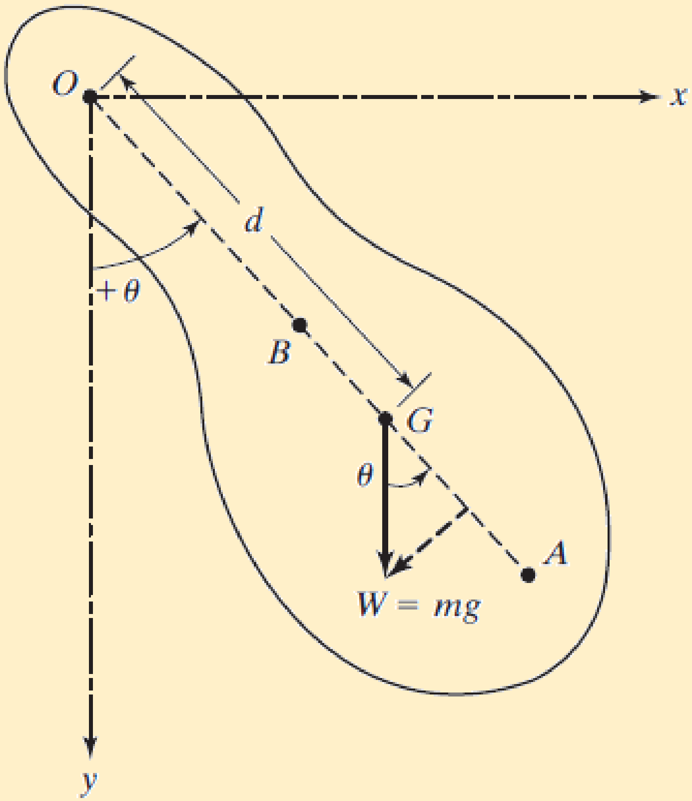
❑Example 2.3

Any rigid body pivoted at a point other than its center of mass will oscillate about the pivot point under its own gravitational force. Such a system is known as a compound pendulum (see the Fig). Find the natural frequency of such a system.



Single DoF free vibration system

VIBRATIONS

Solution

the governing equation is found as:

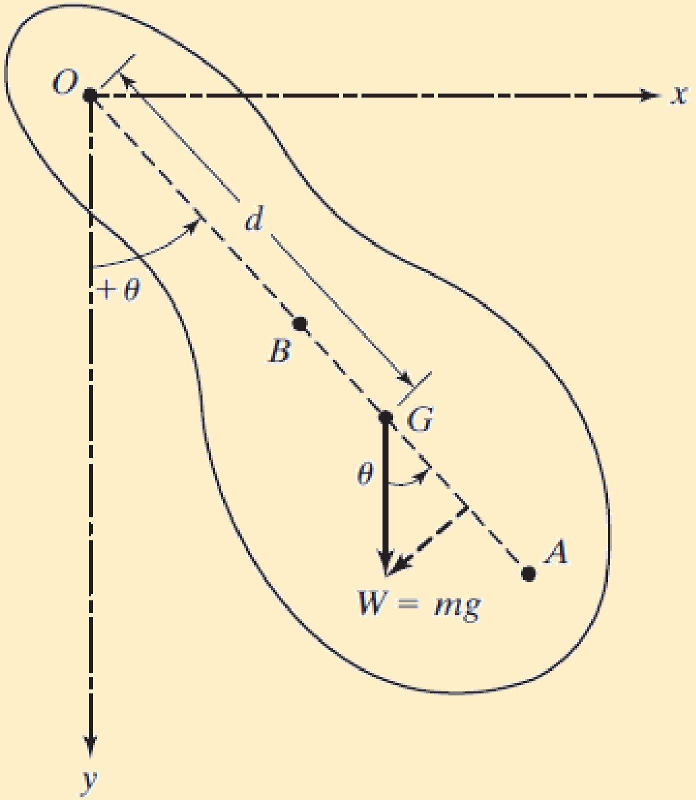
$$J_O\ddot{\theta}+Wd\sin(\theta)=0$$

Assume small angle of vibration:

$$J_O\ddot{\theta}+(Wd)\theta=0$$

So:

$$\omega_n=\sqrt{\frac{Wd}{J_O}}=\sqrt{\frac{mgd}{J_O}}$$



Single DoF free vibration system

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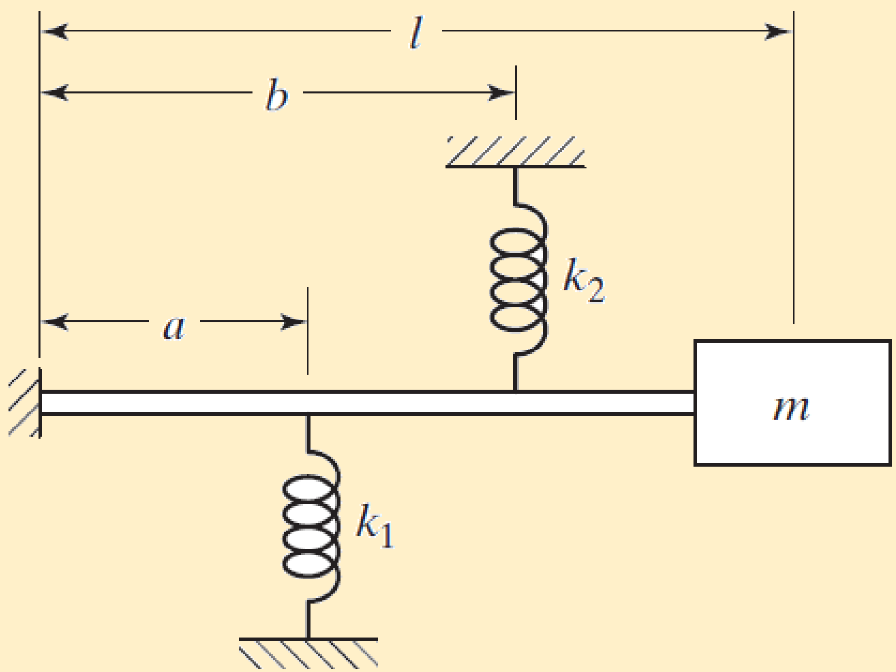
❑Example 2.4: Q2.12

Find the natural frequency of the system shown in Fig. with the springs k1 and k2 in the end of the elastic beam.

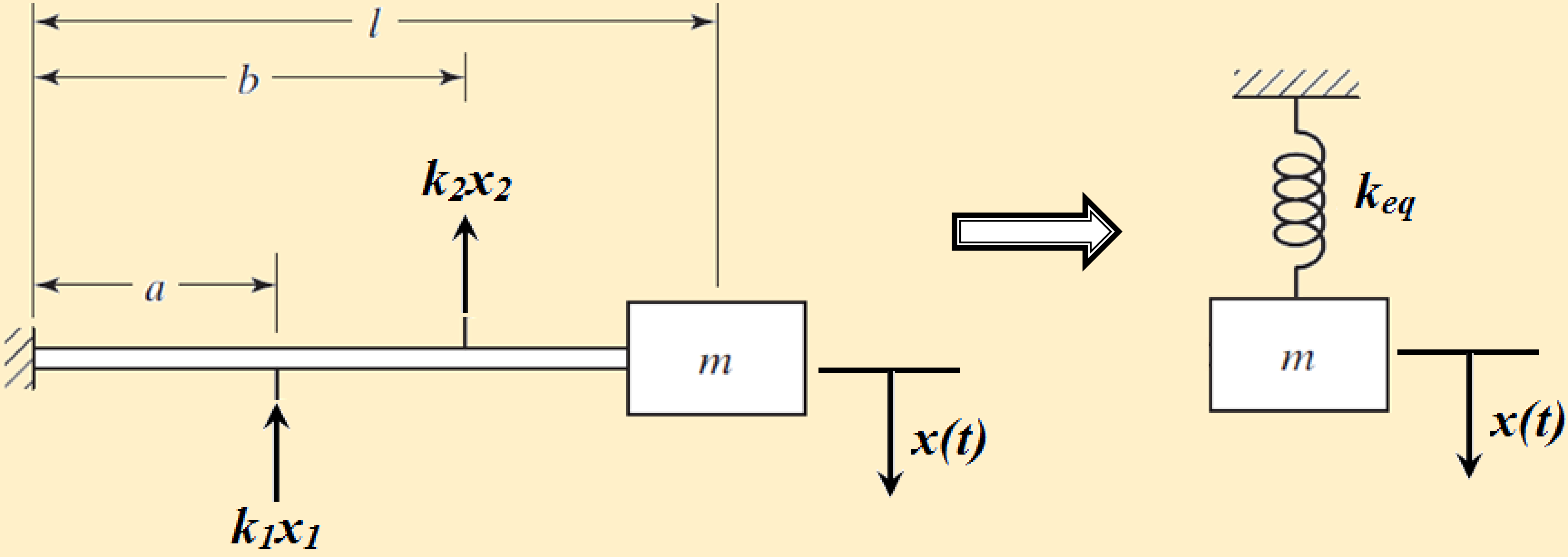


Single DoF free vibration system

VIBRATIONS

Example 2.4: Q2.12

Solution: F.B,D



Single DoF free vibration system

- **Example 2.4: Q2.12**

Solution:

- **k_{eq} is equivalent stiffness for the combination of k_1, k_2 and k_{beam}**

$$k_{beam} = \frac{3EI}{l^3}$$

- **k_1 and k_2 equivalent: apply energy concept**

$$\frac{1}{2}k_{eq,1,2}x^2 = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 \Rightarrow k_{eq} = k_1\left(\frac{x_1}{x}\right)^2 + k_2\left(\frac{x_2}{x}\right)^2$$

Single DoF free vibration system

VIBRATIONS

- **Example 2.4: Q2.12**

Solution:

- **Finding k_{eq}**

$$\frac{1}{k_{eq}} = \frac{1}{k_{eq,1,2}} + \frac{1}{k_{beam}} \Rightarrow k_{eq} = \frac{k_{eq,1,2}k_{beam}}{k_{eq,1,2} + k_{beam}}$$

Single DoF free vibration system

VIBRATIONS

- **Example 2.4: Q2.12**

Solution:

- **Finding natural frequency**

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_{eq,1,2}k_{beam}}{m\left(k_{eq,1,2} + k_{beam}\right)}}$$

$$\omega_n = \sqrt{\frac{\left(k_1\left(\frac{x_1}{x}\right)^2 + k_2\left(\frac{x_2}{x}\right)^2\right)k_{beam}}{m\left(k_1\left(\frac{x_1}{x}\right)^2 + k_2\left(\frac{x_2}{x}\right)^2 + k_{beam}\right)}}$$

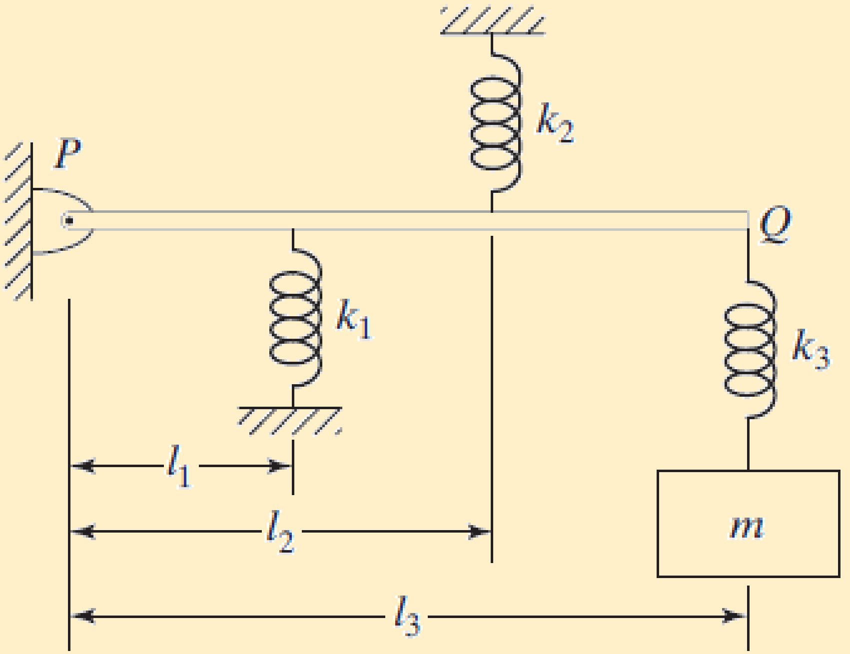
⇒ **x_1, x_2 and x can be found from strength relation**

Single DoF free vibration system

VIBRATIONS

- **Example 2.5: Q2.7**

Three springs and a mass are attached to a rigid, weightless bar *PQ* *as shown in Fig.* Find the natural frequency of vibration of the system.



Single DoF free vibration system

Example 2.5: Q2.7

Solution :

Assume small angular motion $\sin(\theta) \cong \theta$

$$\frac{1}{2}k_{eq,1,2}(\theta l_3)^2 = \frac{1}{2}k_1(\theta l_1)^2 + \frac{1}{2}k_2(\theta l_2)^2 \Rightarrow k_{eq,1,2} = \frac{k_1 l_1^2 + k_2 l_2^2}{l_3^2}$$

Let k_{eq} is the equivalent stiffness for the whole system

$$\frac{1}{k_{eq}} = \frac{1}{k_{eq,1,2}} + \frac{1}{k_3} \Rightarrow k_{eq} = \frac{k_{eq,1,2}k_3}{k_3 + k_{eq,1,2}}$$

Single DoF free vibration system

- **Example 2.5: Q2.7**

Solution :

Now find the natural frequency

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 k_2 l_1^2 + k_2 k_3 l_2^2}{m\left(k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2\right)}}$$

Single DoF free vibration system

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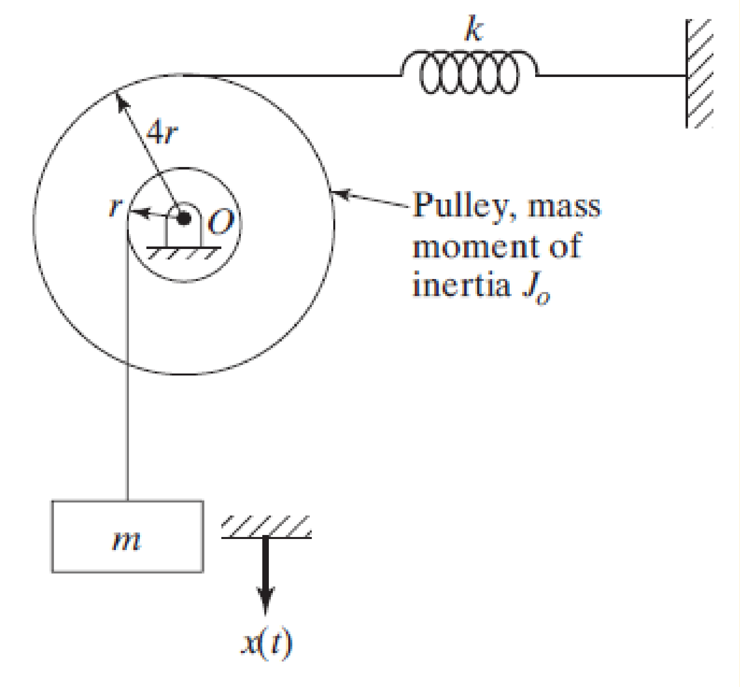
Example 2.5: Q2.45

Draw the free-body diagram and derive the equation of motion using Newton s second law of motion for each of the systems shown in Fig



Single DoF free vibration system

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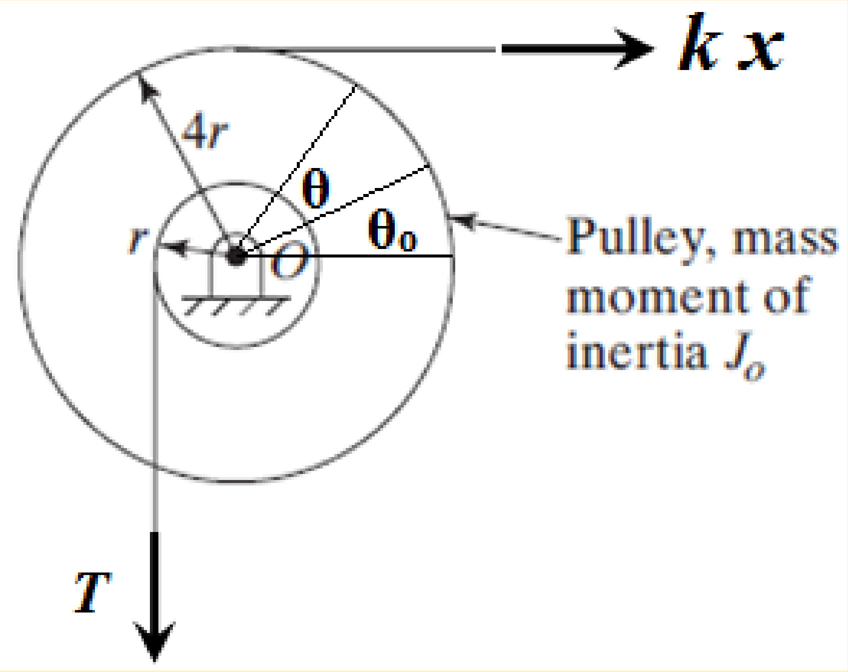
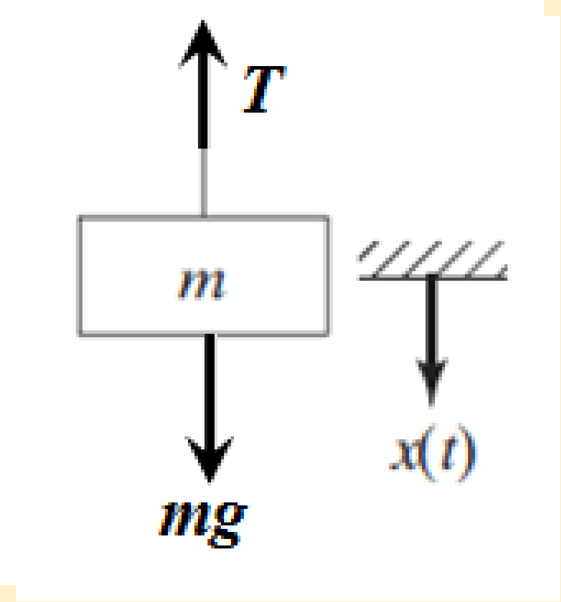
- **Example 2.5: Q2.45**

Solution F.B.D





Single DoF free vibration system

V I B R A T I O N S

Example 2.5: Q2.45

Equation of motion:

The distance: $x = 4r(\theta + \theta_o)$

For mass m: $mg - T = m\ddot{x}$ **--- (1)**

For pulley J_o: $J_o\ddot{\theta} = Tr - 4rk(\theta + \theta_o)(4r)$ **--- (2)**

According to static equilibrium: $mgr = k(4r)(4r)\theta_o \Rightarrow \theta_o = \dfrac{mg}{16rk}$ **--- (3)**

- **Example 2.5: Q2.45**

Equation of motion [cont]:

Substitute equations 1 and 3 into equation 1:

$$J_o\ddot{\theta} = \left(mg - m\ddot{x}\right)r - 16kr^2\left(\theta + \frac{mg}{16rk}\right)$$

$$J_o\ddot{\theta} - \cancel{mgr} + m\ddot{x}r + 16kr^2\theta + \cancel{mgr} = 0 \Rightarrow J_o\ddot{\theta} + m\ddot{x}r + 16kr^2\theta = 0$$

Use the relation $x = r\theta \Rightarrow \ddot{x} = r\ddot{\theta}$ **to relate the translational motion with the rotational one:**

$$\left(J_o + mr^2\right)\ddot{\theta} + \left(16kr^2\right)\theta = 0$$

Single DoF free vibration system

S
T
A
T
I
C
S

End of chapter2 – part I